

An unavoidable case analysis

Archiving old manuscripts, I found at the end of EWD766 "An educational stupidity" of more than 20 years ago the following exercise:

"Prove that none of the decimal numbers 1001, 1001001, 1001001001, 1001001001001, ... is prime." [It is not clear why I underlined "none". EWD]

Here is a proof. Denoting by $k^* \langle \text{string} \rangle^*$ the concatenation of k copies of the digit string enclosed, we deal with the decimal numbers $k^* 001^*$ for $k \geq 2$

- (i) If $k \bmod 3 = 0$, the number is, according to the traditional 3-test, divisible by 3 because the sum of its decimal digits, which equals k , is divisible by 3.
- (ii) If $k \bmod 3 \neq 0$, we see by generalizing the traditional 9-test to the $k^* 9^*$ -test, that the number reduced modulo $k^* 9^*$ equals $k^* 1^*$. Hence the number is divisible by $k^* 1^*$ (because $k^* 9^*$ is). (Note that this observation does not exclude primality for $k=1$.)

In argument (i), the crux is the validity

of the 3-test, which is valid whenever $\text{base} \bmod 3 = 1$, a condition that base 10 (= ten) satisfies. The conclusion has nothing to do with the accident that the length of the repeated string happens to be 3: for $k \bmod 3 = 0$, also $k * 0001^*$ is divisible by 3.

In argument (ii) we generalized the 9-test because the problem was about decimal numbers, but for any base B ($B \geq 2$) there is a $(B-1)$ -test, and $k * 1^*$ divides $k * (B-1)^*$. The conclusion that the number reduced modulo $k * (B-1)^*$ yields $k * 1^*$, however, depends on the fact that k has no factor in common with the length of the iterated string. Argument (ii) is base independent: interpreting $k * 001^*$ and $k * 1^*$ in binary, we find for instance that for $k=4$ and $k=5$, 585 is divisible by 15 and 4681 by 31.

Argument (i) relies on a relation between k and the base of the number system, (ii) on a relation between k and the length of the iterated string.

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