

The arithmetic and geometric means once more

In the following, x, y, c are positive.

In EWD1140, I used

$$(0) \quad (x+y)^2 = (x-y)^2 + 4 \cdot x \cdot y$$

to argue that

$$(1) \quad x, y := c, x+y-c,$$

which does not change $x+y$, increases $x \cdot y$ provided c lies between the initial values of x and y .

In EWD1171, I used (0) to argue that

$$(2) \quad x, y := c, x \cdot y / c,$$

which does not change $x \cdot y$, decreases $x+y$ provided c lies between the initial values of x and y .

In both cases the use of (0) came a little bit as a rabbit and the link between the condition on c and the decrease of the distance between x and y remained informal. Last Thursday, when I asked for an expression that contained both $x+y$ and $x \cdot y$, my

An Thai Nguyen suggested that we look at $(c-x) \cdot (c-y)$, and this expression indeed plays a central role in the derivations from which all rabbits have been removed.

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We want to change x, y such that (i) their sum is not changed, and (ii) their product is increased. Any assignment satisfying (i) can be written like (1); in order to satisfy (ii) we now observe for any c

$$\begin{aligned}
 & \text{"(1) increases } x \cdot y \text{"} \\
 = & \quad \{ \text{program semantics, (1)} \} \\
 & \quad x \cdot y < c \cdot (x + y - c) \\
 = & \quad \{ \text{algebra} \} \\
 & \quad (c-x) \cdot (c-y) < 0
 \end{aligned}$$

We now consider the change of x, y such that (iii) their product is not changed, and (iv) their sum is decreased. Any assignment satisfying (iii) can be written like (2); in order to satisfy (iv) as well, we observe for any positive c

$$\begin{aligned}
 & \text{"(2) decreases } x+y \text{"} \\
 = & \quad \{ \text{program semantics, (2)} \} \\
 & \quad c + x \cdot y / c < x + y
 \end{aligned}$$

$$\begin{aligned}
&= \{ c > 0 \} \\
&\quad c^2 + x \cdot y < c \cdot (x + y) \\
&= \{ \text{algebra} \} \\
&\quad (c - x) \cdot (c - y) < 0
\end{aligned}$$

So, in both cases, the completely forced calculations lead in exactly the same form to the conclusion that c should lie between the initial values of x and y . The secret is that Nguyen's expression can be rewritten as

$$x \cdot y - c \cdot (x + y - c)$$

and as

$$(c^2 + x \cdot y) - (c \cdot x + c \cdot y),$$

i.e. the difference of two products with equal sums of their factors, and the difference of two sums with equal products of their addenda. I was surprised.

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